

# Optimal Error Bounds for Hermite Interpolation

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## INTRODUCTION AND MAIN RESULTS

Let  $u(x) \in C^4[0, h]$  be given; let  $v_3(x)$  be the unique Hermite interpolation polynomial of degree  $\leq 3$  satisfying

$$v_3(0) = u(0), \quad v_3(h) = u(h), \quad v'_3(0) = u'(0), \quad v'_3(h) = u'(h). \quad (1.1)$$

Ciarlet, Schultz and Varga [2, Theorem 9] have obtained pointwise bounds for the error  $e(x) = v_3(x) - u(x)$  and its derivatives in terms of  $U = \max_{0 \leq x \leq h} |u^4(x)|$ . Their bounds are

$$|e^{(k)}(x)| \leq \frac{h^k (x(h-x))^{2-k} U}{k! (4-2k)!}, \quad k = 0, 1, 2. \quad (1.2)$$

For  $k=0$ , (1.2) is best possible, since equality holds for  $u(x) = x^2(h-x)^2$ , whose Hermite interpolation polynomial is  $v=0$ . In 1967 Birkhoff and Priver [3] obtained following optimal error bounds on the derivatives  $|e^{(k)}(x)|$  in terms of  $U$ .

**THEOREM A (Birkhoff and Priver).** *Let  $u(x) \in C^4[0, 1]$ . Then we have*

$$\begin{aligned} |e'(x)| &\leq \frac{Ux(1-x)(1-2x)}{12}, \quad 0 \leq x \leq \frac{1}{3} \\ &\leq \frac{U}{96} \left[ 16x^3 - 105x^2 + 197x - 162 \right. \\ &\quad \left. + \frac{66}{x} - \frac{13}{x^2} + \frac{1}{x^3} \right], \quad \frac{1}{3} \leq x \leq \frac{1}{2}, \end{aligned} \quad (1.3)$$

$$\begin{aligned}
|e''(x)| &\leq \frac{U}{12(1-2x)^3} [48x^5 + 42x^4 - 100x^3 \\
&\quad + 54x^2 - 12x + 1], \quad 0 \leq x \leq \frac{1}{3} \\
&\leq \frac{U}{12} \left[ \frac{1}{2} - 6 \left( x - \frac{1}{2} \right)^2 \right], \quad \frac{1}{3} \leq x \leq \frac{1}{2},
\end{aligned} \tag{1.4}$$

$$|e'''(x)| \leq U \left[ \frac{3}{16} + \frac{3}{2} \left( x - \frac{1}{2} \right)^2 - \left( x - \frac{1}{2} \right)^4 \right], \quad 0 \leq x \leq 1. \tag{1.5}$$

For  $\frac{1}{2} \leq x \leq 1$  the bounds of  $e^{(k)}(x)$  are given by

$$e^{(k)}(x) = e^k(1-x), \quad k = 0, 1, 2, 3. \tag{1.6}$$

Further, from (1.2) ( $k=0$ ), (1.3)–(1.6) the uniform error bounds are given by

$$\begin{aligned}
|e^{(r)}(x)| &\leq \alpha_r U, \quad r = 0, 1, 2, 3; \quad \alpha_0 = \frac{1}{4^2 4!}, \\
\alpha_1 &= \frac{3}{216}, \quad \alpha_2 = \frac{1}{12}, \quad \alpha_3 = \frac{1}{2}.
\end{aligned} \tag{1.7}$$

The object of this paper is to supplement the above theorem by proving the following theorems.

**THEOREM 1.** *Let  $u(x) \in c^3[0, 1]$  and  $v_3(x)$  be the unique, cubic Hermite interpolation polynomial satisfying (1.1) (with  $h=1$ ). Let  $L = \max_{0 \leq x \leq 1} |f'''(x)|$ . Then the following estimates are valid:*

$$\begin{aligned}
|v_3(x) - u(x)| &\leq \frac{4x^2(1-x)^3 L}{3(3-2x)^2}, \quad 0 \leq x \leq \frac{1}{2} \\
|v'_3(x) - u'(x)| &\leq \frac{(2-3x)^3 x L}{27(1-x)^2}, \quad 0 \leq x \leq \frac{1}{3} \\
&\leq \left[ \frac{(2-3x)^3 x}{27(1-x)^2} + \frac{(3x-1)^3 (1-x)}{27x^2} \right] L, \quad \frac{1}{3} \leq x \leq \frac{1}{2}
\end{aligned} \tag{1.8}$$

$$|v''_3(x) - u''(x)| \leq 4x^2(1-x)^2 L, \quad \frac{1}{2} \leq x \leq \frac{2}{3} \tag{1.9}$$

$$\leq \left[ 4x^2(1-x)^2 + \frac{8}{27} \frac{(3x-2)^3}{(2x-1)^2} \right] L, \quad \frac{2}{3} \leq x \leq 1, \tag{1.10}$$

$$|v'''_3(x) - u'''(x)| \leq \max_{0 \leq x, t \leq 1} |u'''(t) - u'''(x)|. \tag{1.11}$$

Moreover (1.8)–(1.10) remain valid if  $L$  is replaced by  $\max_{0 \leq x, t \leq 1} |f'''(t) - f'''(x)|$ . Further, from (1.8)–(1.10) the uniform error bound is given by

$$\begin{aligned} |v_3^{(r)}(x) - u^{(r)}(x)| &\leq \alpha_r L, \quad \alpha_0 = \frac{1}{96}, \\ \alpha_1 &= \frac{13\sqrt{13}-46}{27}, \quad \alpha_2 = \frac{8}{27}, \quad \alpha_3 = 2. \end{aligned} \quad (1.12)$$

**THEOREM 2.** Let  $u(x) \in C^2[0, 1]$  and  $v(x)$  be the unique cubic Hermite interpolation polynomial satisfying (1.1) (with  $h=1$ ). Let  $M = \max_{0 \leq x \leq 1} |f''(x)|$ . Then we have

$$|v_3(x) - u(x)| \leq \frac{4x^2(1-x)^2 M}{(1+2x)(3-2x)}, \quad 0 \leq x \leq 1, \quad (1.13)$$

$$\begin{aligned} |v_3'(x) - u'(x)| &\leq \frac{x(6x^2-9x+4)^2 M}{6(1-x)}, \quad 0 \leq x \leq \frac{1}{3} \\ &\leq \left[ \frac{x(6x^2-9x+4)^2}{6(1-x)} + \frac{(1-x)(1-3x)^2}{6x} \right] M, \\ &\quad \frac{1}{3} \leq x \leq \frac{1}{2}. \end{aligned} \quad (1.14)$$

Moreover, (1.13) and (1.14) remain valid if we replace  $M$  by  $M_1 = \max_{0 \leq x, t \leq 1} |u''(x) - u''(t)|$ . Also,

$$\begin{aligned} |v_3''(x) - u''(x)| &\leq \frac{M_1}{3(1-2x)} [(1-3x)^2 + (2-3x)^2], \quad 0 \leq x \leq \frac{1}{3} \\ &\leq M_1, \quad \frac{1}{3} \leq x \leq \frac{1}{2}. \end{aligned} \quad (1.15)$$

Further, from (1.13)–(1.14) the uniform error bound is given by ( $0 \leq x \leq 1$ )

$$|v_3^{(r)}(x) - u^{(r)}(x)| \leq \beta_r M, \quad \beta_0 = \frac{1}{16}, \beta_1 = 0.251497657 \quad (1.16)$$

and

$$|v_3''(x) - u''(x)| \leq \frac{5}{3} M_1.$$

## 2. PRELIMINARIES

It is known that the unique Hermite interpolation polynomial  $v_3(x)$  satisfying (1.1) (with  $h = 1$ ) is given by

$$\begin{aligned} v_3(x) = & u(0)(1-x)^2(1+2x) + u(1)x^2(3-2x) \\ & + u'(0)x(1-x)^2 + u'(1)x^2(x-1). \end{aligned} \quad (2.1)$$

Using the well-known Peano theorem (3), we may write for  $u(x) \in C^3[0, 1]$

$$v_3^{(r)}(x) - u^{(r)}(x) = \int_0^1 G_3^{(r,0)}(x, t) u^{(3)}(t) dt, \quad r = 0, 1, 2, \quad (2.2)$$

where

$$G^{(i,j)}(x, t) = \frac{\partial^{i+j} G(x, t)}{\partial x^i \partial t^j}, \quad (2.3)$$

$$\begin{aligned} 2G_3^{(0,0)}(x, t) = & t(1-x)^2 [2x - (1+2x)t], & t \leq x \\ = & x^2(1-t)[1 + (2x-3)t], & t \geq x, \end{aligned} \quad (2.4)$$

$$\begin{aligned} G_3^{(1,0)}(x, t) = & t(1-x)[1 - 3x + 3tx], & t \leq x \\ = & x(1-t)[1 - 3t(1-x)], & t > x, \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} G_3^{(2,0)}(x, t) = & t[6x(1-t) + 3t - 4], & t \leq x \\ = & (1-t)[6xt + 1 - 3t], & t > x. \end{aligned} \quad (2.6)$$

Similarly, on using the Peano theorem again [3], we may write for  $u \in C^2[0, 1]$

$$v_3^{(r)}(x) - u^{(r)}(x) = \int_0^1 u^{(2)}(t) G_2^{(r,0)}(x, t) dt, \quad r = 0, 1, \quad (2.7)$$

$$v_3''(x) = -2 \int_0^1 u''(t) [3x - 2 + 3t(1-2x)] dt, \quad (2.7a)$$

$$v_3''(x) - u''(x) = -2 \int_0^1 [u''(t) - u''(x)] [3x - 2 + 3t(1-2x)] dt, \quad (2.7b)$$

where

$$\begin{aligned} G_2^{(0,0)}(x, t) = & (1-x)^2 [x - t(1+2x)], & t \leq x \\ = & x^2 [t(3-2x) - (2-x)], & t \geq x, \end{aligned} \quad (2.8)$$

$$\begin{aligned} G_2^{(1,0)}(x, t) = & (1-x)[6tx + 1 - 3x], & t \leq x \\ = & x[6t(1-x) - (4-3x)], & t \geq x. \end{aligned} \quad (2.9)$$

## 3. PROOF OF THEOREM 1

Let  $\frac{1}{2} \leq x \leq 1$  and note that

$$\alpha_0 = \alpha_0(x) = \frac{2x}{1+2x} \leq x; \quad \beta_0 = \beta_0(x) = \frac{1}{3-2x} \leq x. \quad (3.1)$$

From (2.4) and (3.1) it follows that for  $\frac{1}{2} \leq x \leq 1$

$$\begin{aligned} \int_0^x |G_3^{(0,0)}(x, t)| dt &= \frac{(1-x)^2(1+2x)}{2} \left[ \int_0^{x_0} t(\alpha_0 - t) dt + \int_{x_0}^x t(t - \alpha_0) dt \right] \\ &= \frac{1}{3} \frac{x^3(3-3x+4x^3)(1-x)^2}{(1+2x)^2}. \end{aligned} \quad (3.2)$$

Again, on using (2.4) and (3.1) it follows that for  $\frac{1}{2} \leq x \leq 1$

$$\int_x^1 |G_3^{(0,0)}(x, t)| dt = \frac{x^2(3-2x)}{2} \int_x^1 (1-t)(t - \beta_0) dt = \frac{x^3(1-x)^3}{3}. \quad (3.3)$$

From (3.2) and (3.3) we have

$$\int_0^1 |G_3^{(0,0)}(x, t)| dt = \frac{4x^3(1-x)^2}{3(1+2x)^2}, \quad \frac{1}{2} \leq x \leq 1. \quad (3.4)$$

From symmetry arguments we also have

$$\int_0^1 |G_3^{(0,0)}(x, t)| dt = \frac{4x^2(1-x)^3}{3(3-2x)^2}, \quad 0 \leq x \leq \frac{1}{2}. \quad (3.5)$$

From (3.4), (3.5) and (2.2) follow (1.8). Next we prove (1.9). First, we note that

$$\begin{aligned} \alpha_1 = \alpha_1(x) &= \frac{1}{3(1-x)} \geq x, \quad 0 \leq x \leq \frac{1}{2}; \\ \beta_1 = \beta_1(x) &= \frac{3x-1}{3x} \leq x \quad \text{for} \quad \frac{1}{3} \leq x \leq \frac{1}{2}. \end{aligned} \quad (3.6)$$

Let  $0 \leq x \leq \frac{1}{3}$ . Then, on using (2.5) and (3.6), we obtain

$$\begin{aligned} \int_0^x |G_3^{(1,0)}(x, t)| dt &= (1-x) \int_0^x (t(1-3x) + 3xt^2) dt \\ &= \frac{x^2(1-x)^2(1-2x)}{2} \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} \int_x^1 |G_3^{(1,0)}(x, t)| dt &= 3x(1-x) \left[ \int_x^{\alpha_1} (1-t)(\alpha_1-t) dt \right. \\ &\quad \left. + \int_{\alpha_1}^1 (1-t)(t-\alpha_1) dt \right] \\ &= \frac{x(1-x)}{2} [2(1-\alpha_1)^3 + 2(1-x)^3 - 3(1-x)^2(1-\alpha_1)]. \end{aligned} \quad (3.8)$$

Combining (3.7), (3.8), and using (3.6) we obtain for  $0 \leq x \leq \frac{1}{3}$

$$\int_0^1 |G_3^{(1,0)}(x, t)| dt = \frac{(2-3x)^3 x}{27(1-x)^2}. \quad (3.9)$$

Next, consider  $\frac{1}{3} \leq x \leq \frac{1}{2}$ . Using (2.5), (3, 6) we have

$$\begin{aligned} \int_0^x |G_3^{(1,0)}(x, t)| dt &= 3x(1-x) \left[ \int_0^{\beta_1} (\beta_1-t)t dt + \int_{\beta_1}^x (t-\beta_1)t dt \right] \\ &= x(1-x) \left[ \beta_1^3 + \frac{3}{2}x^2(x-\beta_1) - \frac{x^3}{2} \right]. \end{aligned} \quad (3.10)$$

Similarly

$$\begin{aligned} \int_x^1 |G_3^{(1,0)}(x, t)| dt &= 3x(1-x) \left[ \int_x^{\alpha_1} (\alpha_1-t)(1-t) dt + \int_{\alpha_1}^1 (t-\alpha_1)(1-t) dt \right] \\ &= x(1-x) \left[ (1-\alpha_1)^3 - \frac{(1-x)^3}{2} + \frac{3}{2}(\alpha_1-x)(1-x)^2 \right]. \end{aligned} \quad (3.11)$$

On combining (3.10) and (3.11) we have  $\frac{1}{3} \leq x \leq \frac{1}{2}$

$$\int_0^1 |G_3^{(1,0)}(x, t)| dt = \frac{x(1-x)}{27} \left[ \frac{(3x-1)^3}{x^3} + \frac{(2-3x)^3}{(1-x)^3} \right]. \quad (3.12)$$

From (3.19), (3.12) and (2.2) follows (1.9). Now, we aim to prove (1.10). First, let  $\frac{1}{2} \leq x \leq \frac{2}{3}$ . Then from (2.6) it follows that

$$\begin{aligned} \int_0^1 |G_3^{(2,0)}(x, t)| dt &= \int_0^x t[2(2-3x) + 3t(2x-1)] dt \\ &\quad + \int_x^1 (1-t)[1 + 3t(2x-1)] dt \\ &= 4x^2(1-x)^2. \end{aligned} \quad (3.13)$$

Next we consider  $\frac{2}{3} \leq x \leq 1$  and we note that

$$\alpha_2 = \alpha_2(x) = \frac{2(3x-2)}{3(2x-1)} \leq x. \quad (3.14)$$

Now, on using (2.6) and (3.14) we obtain

$$\begin{aligned} \int_0^x |G_3^{(2,0)}(x, t)| dt &= 3(2x-1) \int_0^x t |(\alpha_2 - t)| dt \\ &= 3(2x-1) \left[ \int_0^{\alpha_2} t(\alpha_2 - t) dt + \int_{\alpha_2}^x t(t - \alpha_2) dt \right] \\ &= \frac{(2x-1)}{2} [2\alpha_2^3 + 2x^3 - 3\alpha_2 x^2]. \end{aligned} \quad (3.15)$$

Further, on using (2.6) and (3.14) we obtain

$$\int_x^1 |G_3^{(2,0)}(x, t)| dt = \int_x^1 (1-t)[1+3t(2x-1)] dt = 2x^2(1-x)^2. \quad (3.16)$$

From (3.15), (3.16) we obtain for  $\frac{2}{3} \leq x \leq 1$

$$\int_0^1 |G_3^{(2,0)}(x, t)| dt = 4x^2(1-x)^2 + \frac{8}{27} \frac{(3x-2)^3}{(2x-1)^2}. \quad (3.17)$$

From (3.13) and (3.17) we prove (1.10). It remains to prove (1.11).

For this purpose we note that

$$v_3'''(x) = 6 \int_0^1 t(1-t) u'''(t) dt, \quad 1 = 6 \int_0^1 t(1-t) dt.$$

Therefore

$$v_3'''(x) - u'''(x) = 6 \int_0^1 t(1-t)(u'''(t) - u'''(x)) dt. \quad (3.18)$$

From (3.18), (1.11) follows at once. From

$$\int_0^1 G_3^{(r,0)}(x, t) dt = 0, \quad r = 0, 1, 2, \quad (3.19)$$

it follows that for  $r = 0, 1, 2$

$$v_3^{(r)}(x) - u^{(r)}(x) = \int_0^1 G_3^{(r,0)}(x, t)[u'''(t) - u'''(x)] dt.$$

Therefore

$$|v_3^{(r)}(x) - u^{(r)}(x)| \leq \max_{0 \leq x, t \leq 1} |u'''(t) - u'''(x)| \cdot \int_0^1 |G_3^{(r,0)}(x, t)| dt.$$

Now, using (3.4), (3.5), (3.9), (3.12), (3.13) and (3.17), the same estimates hold in (1.8), (1.9) and in (1.10) provided  $L$  is replaced by  $\max_{0 \leq x, t \leq 1} |f'''(t) - f'''(x)|$ . This proves Theorem 1.

The proof of Theorem 2 is similar to Theorem 1, so we omit the details.

#### 4. REMARKS

Theorems 1 and 2 are best possible. Let us consider the function

$$\begin{aligned} f_n(x) &= \left(1 - \frac{2}{n}\right)x^2 - \frac{4}{3}x^3 + \frac{6n-4}{3n^2}x + \frac{4n-3n^2-2}{6n^2}, \\ 0 \leq x &\leq \frac{1}{2} - \frac{1}{n} \\ &= \frac{n(2x-1)^4}{48} - \frac{(2x-1)^2}{4} + \frac{1}{12}, \quad \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ &= \left(1 - \frac{2}{n}\right)(1-x)^2 - \frac{4}{3}(1-x)^3 + \frac{(6n-4)}{3n^2}(1-x) \\ &\quad + \frac{4n-3n^2-2}{6n^3}, \quad \frac{1}{2} + \frac{1}{n} \leq x \leq 1. \end{aligned}$$

It is easy to see that  $f_n(x) \in C^{(3)}[0, 1]$  and  $\max_{0 \leq x \leq 1} |f_n'''(x)| = 8$ . If  $v_3(f_n, x)$  denotes the corresponding cubic Hermite interpolation polynomial then one obtains

$$\left|f_n\left(\frac{1}{2}\right) - v_3\left(f_n, \frac{1}{2}\right)\right| = \frac{1}{96} - \frac{1}{24n^2} + \frac{1}{24n^3}.$$

Therefore (1.12) is best possible for  $r = 0$ . Next, we will prove that (1.12) is best possible for  $r = 1$ . Let us consider

$$\begin{aligned} g_n(x) &= \frac{x^3}{6} + \frac{A}{2}x^2, \quad 0 \leq x \leq \alpha - \frac{1}{n} \\ &= -\frac{n}{24}(x-\alpha)^4 + \frac{1}{2}\left(\alpha - \frac{1}{2n} + A\right)(x-\alpha)^2 + B(x-\alpha) + C, \\ \alpha - \frac{1}{n} &\leq x \leq \alpha + \frac{1}{n} \end{aligned}$$



$$= \frac{-x^3}{6} + (2\alpha + A) \frac{x^2}{2} + \left( \frac{1}{2} - 2\alpha - A \right) x + D,$$

$$\alpha + \frac{1}{n} \leq x \leq 1,$$

where

$$\alpha = \frac{\sqrt{13}-1}{6}, \quad \beta = \frac{5-\sqrt{13}}{6}, \quad A = \alpha^2 - 2\alpha + \frac{1}{2} + \frac{1}{3n^2},$$

$$B = \frac{\alpha^2}{2} + A\alpha - \frac{1}{6n^2}, \quad C = \frac{\alpha^3}{6} + \frac{A}{2} \alpha^2 - \frac{1}{24n^3},$$

$$D = \frac{\alpha^3}{3} + \frac{\alpha}{3n^2}.$$

Again, it is easy to see that  $g_n(x) \in C^{(3)}[0, 1]$ ,  $\max_{0 \leq x \leq 1} |g_n'''(x)| = 1$ .

$$\lim_{n \rightarrow \infty} |v'_3(g_n, \beta) - g'_n(\beta)| = \frac{13\sqrt{13}-46}{27},$$

where  $\beta$  is defined above. This proves that (1.12) cannot be improved in the case  $r = 1$  as well. Next, we consider

$$\begin{aligned} h_n(x) &= \frac{x^3}{6} + \frac{x^2}{2} \left( -\frac{7}{18} + \frac{1}{2n} \right) + x \left( \frac{1}{6n^2} - \frac{1}{3n} \right) + A_1, \\ 0 &\leq x \leq \frac{2}{3} - \frac{1}{n} \\ &= -\frac{n}{24} \left( x - \frac{2}{3} \right)^4 + \frac{5}{36} \left( x - \frac{2}{3} \right)^2 - \frac{1}{27} \left( x - \frac{2}{3} \right) - \frac{1}{27}, \\ \frac{2}{3} - \frac{1}{n} &\leq x \leq \frac{2}{3} + \frac{1}{n} \\ &= \frac{-x^3}{6} + \left( \frac{17}{18} + \frac{1}{2n} \right) \frac{x^2}{2} - \left( \frac{4}{9} + \frac{1}{3n} + \frac{1}{6n^2} \right) x + A_2, \\ \frac{2}{3} + \frac{1}{n} &\leq x \leq 1, \end{aligned}$$

where

$$A_1 = \frac{1}{9n} - \frac{1}{9n^2} + \frac{1}{24n^3}, \quad A_2 = \frac{8}{81} + \frac{1}{9n} + \frac{1}{9n^2} + \frac{1}{24n^3}.$$

Again  $h_n(x) \in C^{(3)}[0, 1]$ ,  $\max_{0 \leq x \leq 1} |h_n'''(x)| = 1$ . Also note that

$$|v_3''(h_n, 1) - h_n''(1)| = \frac{8}{27} - \frac{2}{3n^2}.$$

Thus (1.12) is also best possible for  $r = 2$  as well. In a similar way it can be shown that (1.12) cannot be improved for  $r = 3$  as well. We omit the details. It may be worthwhile to mention that in the limit as  $f_{n \rightarrow \infty}(x)$ , tends to a perfect spline function. Similar statement is also true for  $g_n(x)$  and  $h_n(x)$ .

In an analogous way it can be shown that Theorem 2 is also best possible.

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